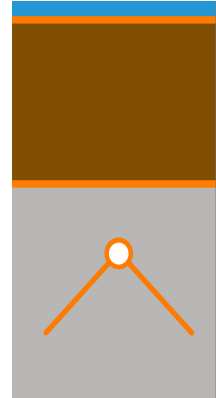


# Temperature distribution above and below a heated layer containing radioactive waste

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In an ART-TEL 2.1 facility for the final disposal of nuclear waste, radioactive waste and high-level waste, as proposed by Ing. Goebel, horizontal boreholes with a diameter of  $D = 16.4$  m are drilled to a depth of 700 m. From these tubes, numerous boreholes extend downwards at an angle of  $45^\circ$ . Their length is 337m, with the first 10 m not being filled with containers. The containers are spaced 1.314 m apart within the boreholes and between the boreholes, and initially each emits 4.31 W of heat. The heat output is  $q_0 = 4.31 \text{ W} / 1.3142 \text{ m}^2 = 2.4962 \text{ W/m}^2$ . As the heat flows away on both sides of the plane of the boreholes, the calculation is such that heat diffusion into the semi-infinite space occurs with  $q_0 = 1,2481 \text{ W/m}^2$  as the boundary condition, with an exponential decrease over time corresponding to radioactive decay.  $q(t) = q_0 * \exp(-b*t)$ . The heat output of all nearly 19 million containers is 81.68 MW.



The following differential equation applies to heat diffusion:  $\frac{\delta}{\delta t} T(x, t) = a * \frac{\delta^2}{\delta x^2} T(x, t)$

with the boundary condition:  $-\lambda * \frac{\delta}{\delta x} T(x=0, t) = q(t) = q_0 * \exp(-b*t)$

The solution is best obtained using the Laplace transform:

$$s * L(T(x, t), t, s) - T(x, 0) = a * \left( \frac{\delta^2}{\delta x^2} L(T(x, t), t, s) \right)$$

If we set  $T(x, 0) = 0$ , the solution is  $U(x, s) = \exp\left(\frac{\sqrt{(s)} * x}{\sqrt{(a)}}\right) * F_2(s) + \exp\left(\frac{-\sqrt{(s)} * x}{\sqrt{(a)}}\right) * F_1(s)$

Since the solution must be bounded as  $x$  approaches infinity, it follows that  $F_2(s) = 0$ .

The transformed boundary condition gives  $-\lambda * \frac{\delta}{\delta x} U(x, s) = \frac{q_0}{s+b}$

The derivative of the solution  $U(x, s)$  yields:  $-\lambda * \frac{\delta}{\delta x} \exp\left(\frac{-\sqrt{(s)} * x}{\sqrt{(a)}}\right) * F_1(s) = \frac{q_0}{s+b}$

It follows at the boundary  $x = 0$  that :  $F_1(s) = \frac{q_0 * \exp(0) * \sqrt{(a)}}{\sqrt{(s)} * \lambda * (s+b)}$

Thus, the Laplace transform of the solution is :  $U(x, s) = \left( \frac{q_0 * \sqrt{(a)}}{\sqrt{(s)} * \lambda * (s+b)} \right) * \exp\left(\frac{-\sqrt{(s)} * x}{\sqrt{(a)}}\right)$

With the abbreviations  $C = (q_0 * \sqrt{a} / \lambda)$  and  $k = x / \sqrt{a}$  we get  $U(x, s) = C * \frac{\exp(-\sqrt{(s)} * k)}{\sqrt{(s)} * (s+b)}$

To find the inverse Laplace transform of  $F(s) = \frac{\exp(-k * \sqrt{(s)})}{(s+b) * \sqrt{(s)}}$

we use the property of convolution and known correspondences from the Laplace table.

## Step 1: Identification of the components

We can write  $F(s)$  as the product of two functions  $G(s)$  and  $H(s)$

$$G(s) = \frac{\exp(-k\sqrt{s})}{\sqrt{s}} \quad \text{und} \quad H(s) = \frac{1}{s+b}$$

## Step 2: Inverse Laplace transform of the individual parts

From the standard tables of the Laplace transform, we know:

$$L^{-1}(G(s)) = L^{-1}\left(\frac{\exp(-k\sqrt{s})}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right) = g(t)$$

$$L^{-1}(H(s)) = L^{-1}\left(\frac{1}{s+b}\right) = \exp(-bt) = h(t)$$

## Step 3: Application of the convolution theorem

The convolution theorem states:  $L^{-1}(G(s) * H(s)) = (g * h)(t) = \int_0^t g(\tau) * h(t-\tau) d\tau$

Let us substitute our functions:  $f(t) = (g * h)(t) = \int_0^t \frac{1}{\sqrt{\pi \tau}} \exp\left(-\frac{k^2}{4\tau}\right) \exp(-b(t-\tau)) d\tau$

Let us take the constant  $\exp(-bt)$  from the integral:

$$f(t) = \frac{\exp(-bt)}{\sqrt{\pi}} * \int_0^t \frac{1}{\sqrt{\tau}} \exp\left(-\frac{k^2}{4\tau} + b\tau\right) d\tau$$

## Step 4: Solving the Integral

To solve this integral, a substitution is made (typically  $u = \sqrt{\tau}$ , with  $du = \frac{1}{2\sqrt{\tau}} d\tau$ )

$$f(t) = \frac{2 * \exp(-bt)}{\sqrt{\pi}} * \int_0^{\sqrt{t}} \exp\left(-\frac{k^2}{4u^2} + b * u^2\right) du$$

The exponent in the integral can be expanded

$$-\frac{k^2}{4u^2} + b * u^2 = -\left(\frac{k}{2u} + \sqrt{b} * u\right) * \left(\frac{k}{2u} - \sqrt{b} * u\right) = -\left(\frac{k}{2\sqrt{\tau}} + \sqrt{b * \tau}\right) * \left(\frac{k}{2\sqrt{\tau}} - \sqrt{b * \tau}\right)$$

After applying the standard integrals with partial integration for this specific form, the result is obtained using the error function  $\text{erf}(z)$  or the complementary error function  $\text{erfc}(z)$ .

CAUTION when using artificial intelligence. 3 attempts yielded 3 different incorrect results.

The result, using the abbreviations  $y = x * \sqrt{(b/a)}$  and  $\xi = x / \sqrt{(4 * a * t)}$ , is:

$$f(t, x) = \frac{-I * \exp(-bt)}{2 * \sqrt{(b)}} * (\exp(-I * y) * \text{erfc}(\xi - I * \sqrt{(bt)}) - \exp(I * y) * \text{erfc}(\xi + I * \sqrt{(bt)}))$$

The entire function  $u(s, x) = \frac{q_0 * \sqrt{(a)}}{\lambda} * \frac{\exp(-\sqrt{(s/a)} * x)}{(\sqrt{(s)} * (s+b))}$  is transformed back to

$$T(t, x) = \frac{-I * q_0 * \exp(-bt)}{2 * \sqrt{(b/a)} * \lambda} * (\exp(-I * y) * \text{erfc}(\xi - I * \sqrt{(bt)}) - \exp(I * y) * \text{erfc}(\xi + I * \sqrt{(bt)}))$$

where  $y = x * \sqrt{(b/a)}$  and  $\xi = x / \sqrt{(4 * a * t)}$

The material properties are:

Decay rate  $b=4,588 \cdot 10^{-10} \text{ 1/s}$       Density of the salt  $\rho=2200 \text{ kg/m}^3$   
 Specific heat capacity  $c_p=1200 \text{ J/(kg} \cdot \text{K)}$       Thermal conductivity  $\lambda=5,4 \text{ W/(m} \cdot \text{K)}$   
 Thermal conductivity  $a=\lambda/(\rho \cdot c_p)=2,045454546 \cdot 10^{-6} \text{ m}^2/\text{s}$

Thus, the spatial and temporal temperature distribution is given by:

$$T(x, t) = -I \cdot 7,71642253 \cdot \exp(-4,588 \cdot 10^{-10} \cdot t) \cdot \left( \exp(-I \cdot 0,01497672268 \cdot x) \cdot \operatorname{erfc}\left(349,6029494 \cdot x / \sqrt{t} - I \cdot 0,2141961718 \cdot 10^{-4} \cdot \sqrt{t}\right) - \exp(I \cdot 0,01497672268 \cdot x) \cdot \operatorname{erfc}\left(349,6029494 \cdot x / \sqrt{t} + I \cdot 0,2141961718 \cdot 10^{-4} \cdot \sqrt{t}\right) \right)$$

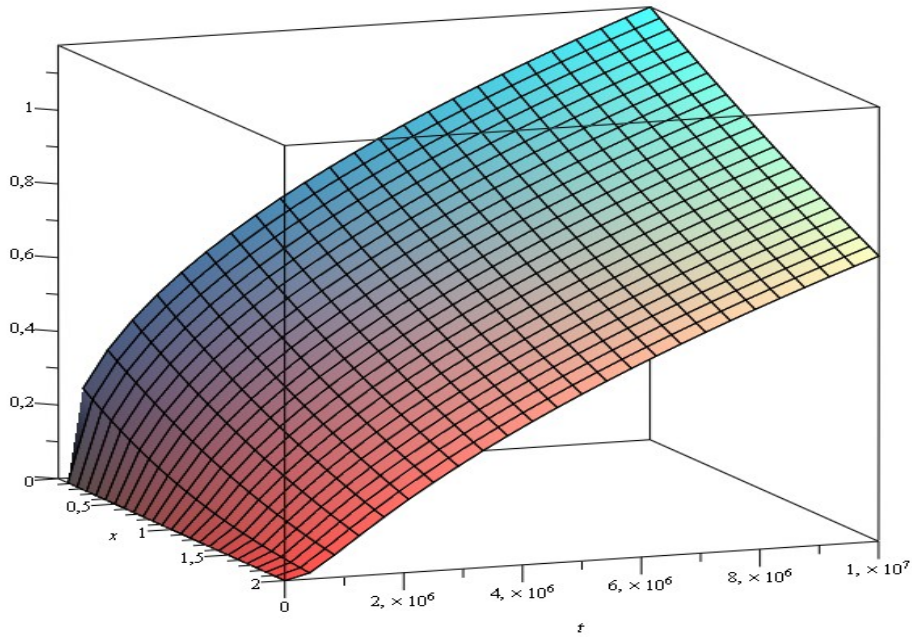


Fig. 1 Temperature field in K, Distance x in m, time t in s  $t < 115$  days

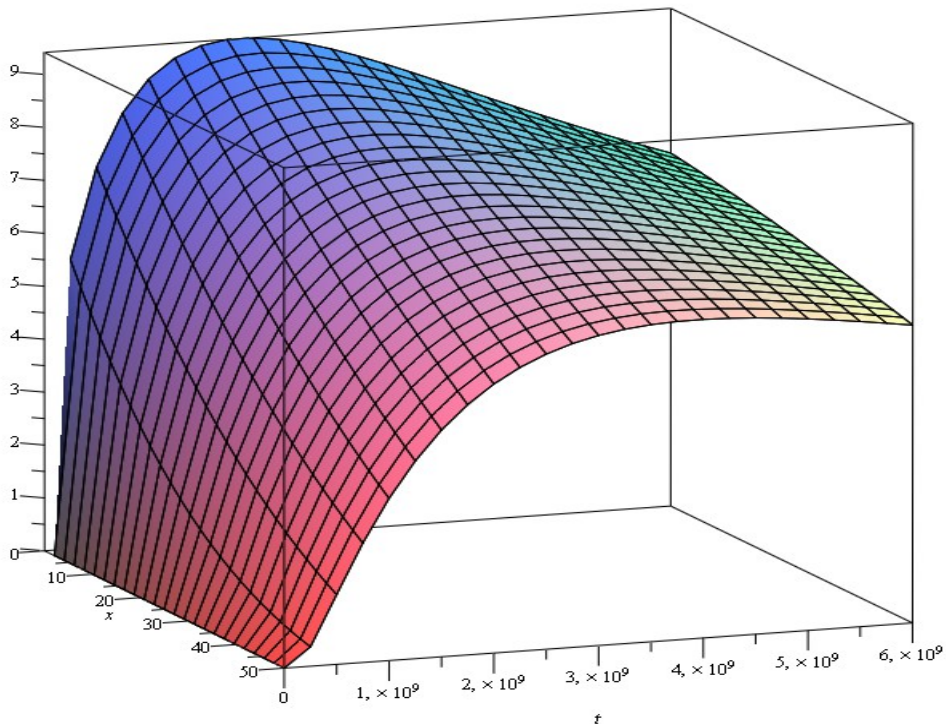


Fig. 2 Temperature field in K, Distance x in m, time t in s  $t < 190$  years

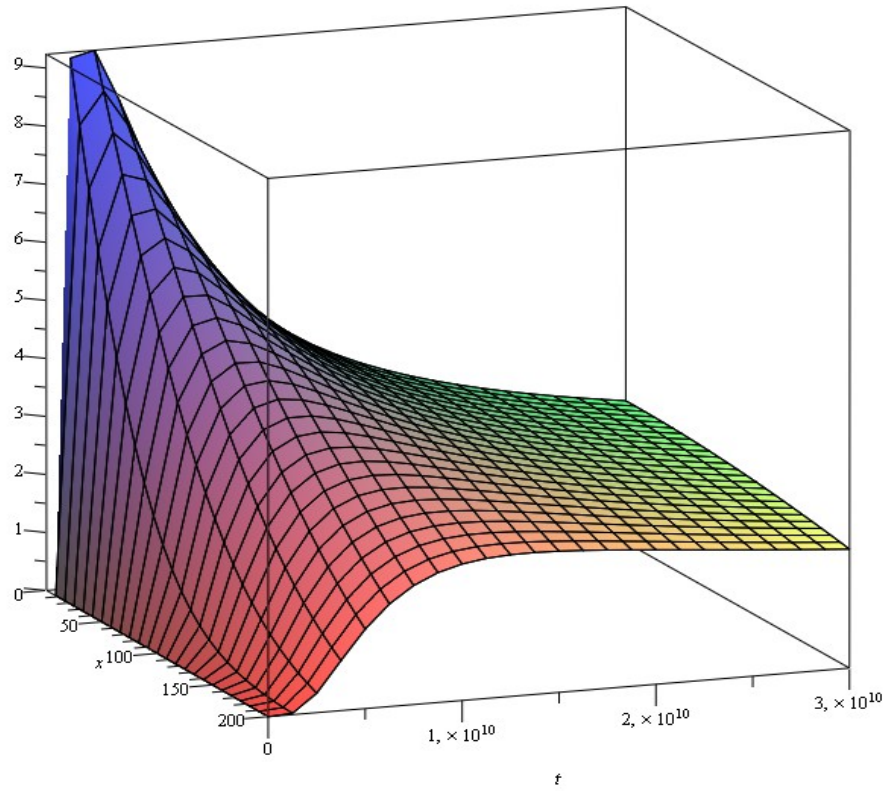


Fig. 3 Temperature field in K, Distance  $x$  in m, time  $t$  in s  $t < 950$  years

After 950 years, the heat is distributed almost evenly over the next 200 m, and the temperature is approximately 2.4 K at the heating level and just under 1 K at a distance of 500 m.

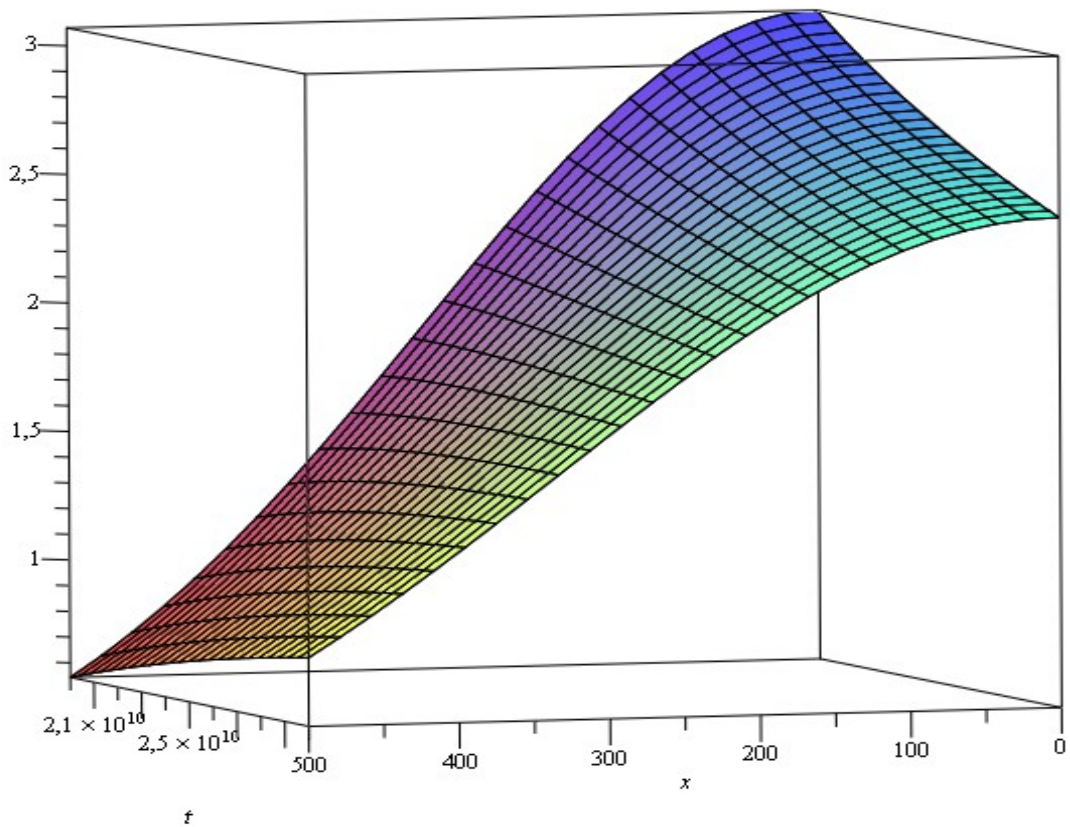


Fig. 4 Temperature field in K, Distance  $x$  in m, time  $t$  in s,  $t < 950$  years

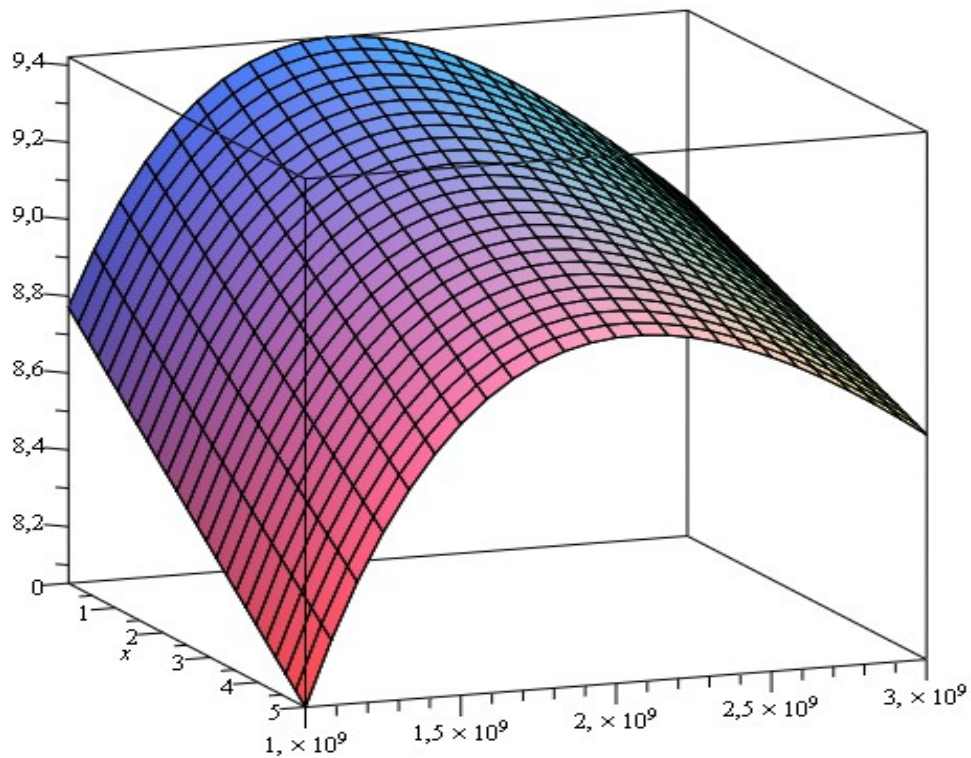


Fig. 5 Temperature field in K, Distance  $x$  in m, time  $t$  in s, maximum at 9,4 K after 58.6 years

The maximum temperature of 9.4 K is reached after approximately 58.6 years at the level of the containers (topmost curve). The curves below represent greater distances of up to 5 m from this level.

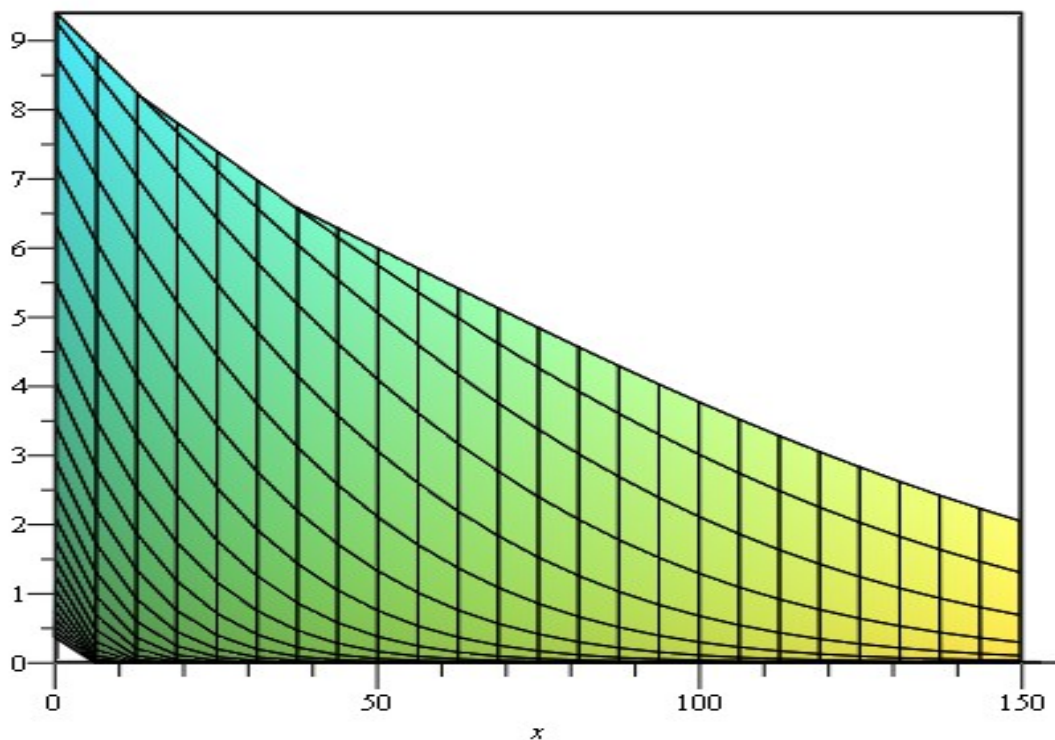


Fig. 6 Temperature field in K, Distance  $x$  in m, time  $t$  in s,  $10^6 \text{ s} < t < 10^{9.6} \text{ s} = 126.2 \text{ years}$

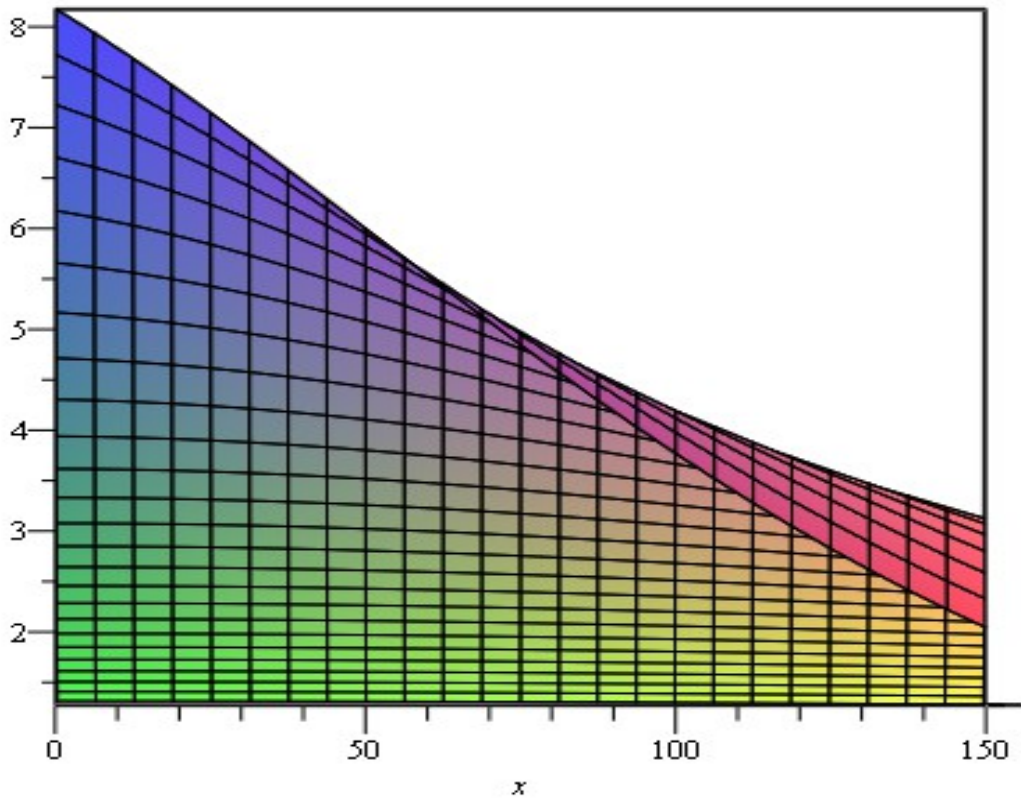


Fig. 7 Temperature field in K, Distance  $x$  in m, time  $t$  in s,  $10^{9.6} \text{ s} < t < 10^{11} \text{ s} = 3170 \text{ years}$

The temperature rises slowly at first until it reaches its maximum after 58.6 years. In fig. 7 line show later times till  $t = 10^{11} \text{ s} = 3170 \text{ years}$ . The heat then penetrates further and further into the salt, and the temperature falls again, even in the layer of the container.

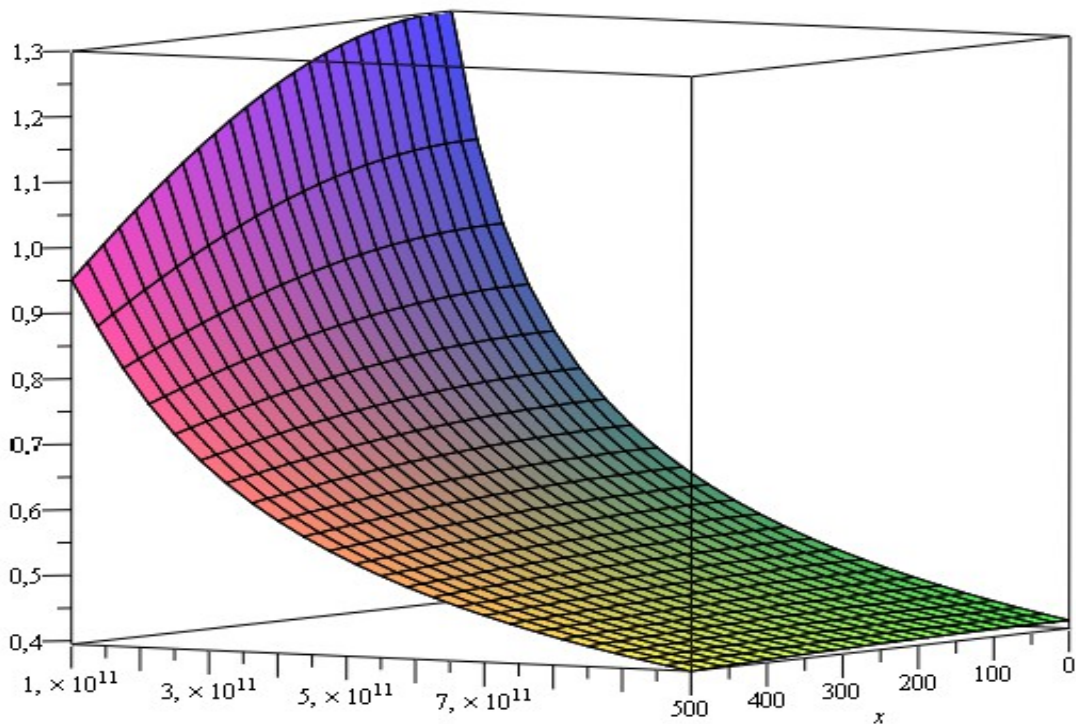


Fig. 8 Temperature field in K, Distance  $x < 500 \text{ m}$ , time  $t$  in s,  $t < 10^{12} \text{ s} = 31688 \text{ years}$

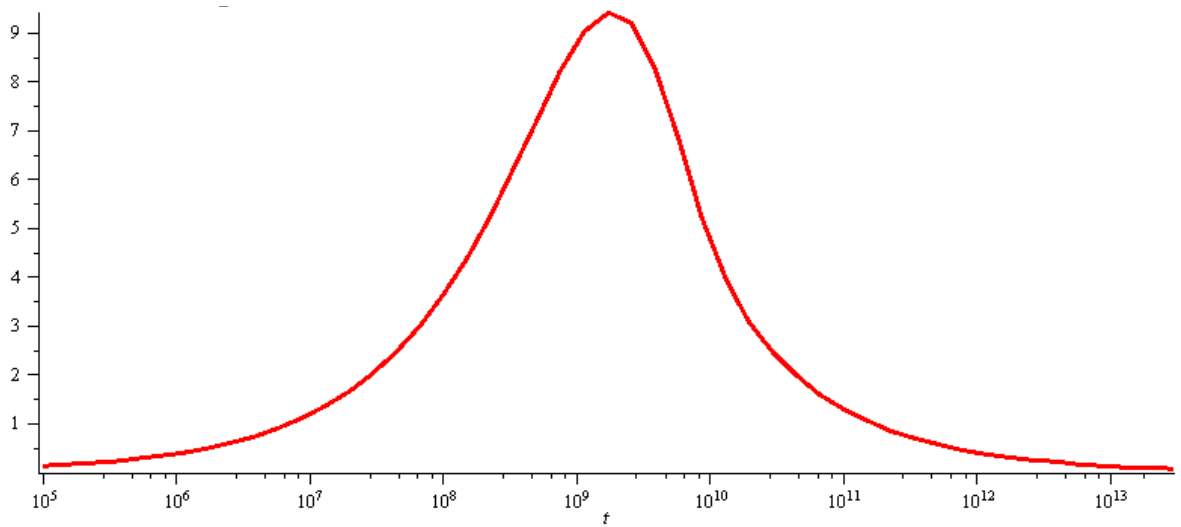


Fig. 9 Temperature at container layer in K, time  $10^5 \text{ s} = 1,16 \text{ days} < t < 3 \cdot 10^{13} \text{ s} = 950642 \text{ years}$ , peak after  $1.85 \cdot 10^9 \text{ s} = 58.6 \text{ years}$

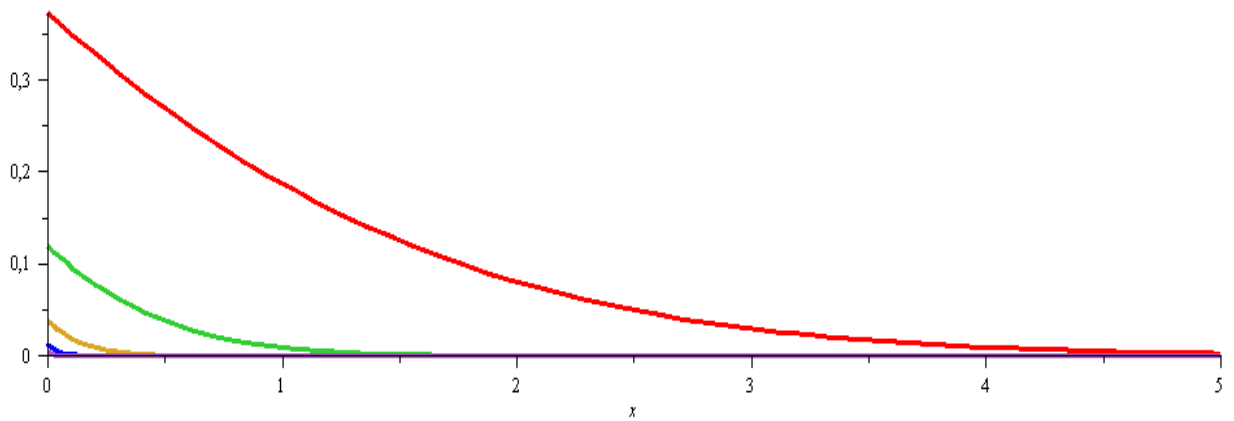


Fig. 10 Temperature in K at distance  $x < 5 \text{ m}$ , time  $t = 1000 \text{ s}, 10^4 \text{ s}, 10^5 \text{ s}, 10^6 \text{ s} = 11.6 \text{ days}$

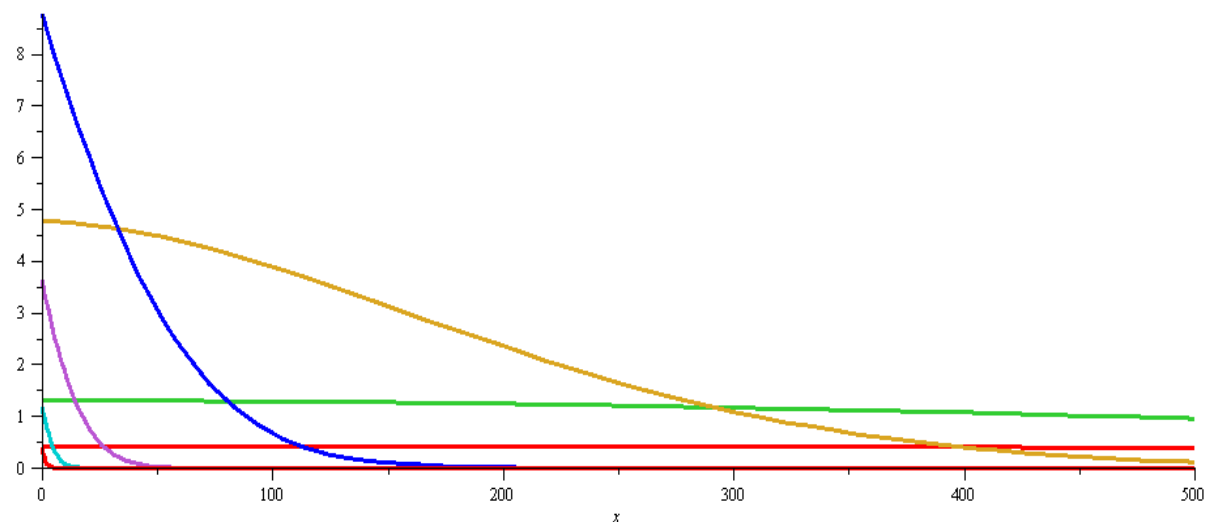


Fig. 11 Temperature in K at distance  $x < 500 \text{ m}$ , time  $t = 10^6 \text{ s}, 10^7 \text{ s}, 10^8 \text{ s}, 10^9 \text{ s}, 10^{10} \text{ s}, 10^{11} \text{ s}, 10^{12} \text{ s} = 31688 \text{ years}$

The incoming heat causes the temperature to rise slowly until, after approximately  $10^9 \text{ s} = 31,7 \text{ years}$ , the heat input is too low to maintain the temperature in the heating plane. The temperature drops again and, after  $10^{12} \text{ s} = 31688 \text{ years}$ , the heat is evenly distributed and the temperature is less than 0.5 K higher everywhere than at the start.