Temperature distribution above and below a heated layer containing radioactive waste

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In an ART-TEL 2.0 facility for the final disposal of nuclear waste, radioactive waste and high-level waste, as proposed by Ing. Goebel, horizontal boreholes with a diameter of D = 16.4 m are drilled to a depth of 700 m. From these tubes, numerous boreholes extend downwards at an angle of 45°. Their length is 337m, with the first 10 m not being filled with containers. The containers are spaced 1.314 m apart within the boreholes and between the boreholes, and initially each emits 4.31 W of heat. The heat output is q_0 =4.31W/1.3142 m² = 2.4962 W/m². As the heat flows away on both sides of the plane of the boreholes, the calculation is such that heat diffusion into the semi-infinite space occurs with q_0 =1,2481 W/m² as the boundary condition, with an exponential decrease over time corresponding to radioactive decay. q(t) = q_0 * exp(-b*t). The heat output of all nearly 19 million containers is 81.68 MW.

The following differential equation applies to heat diffusion: $\frac{\delta}{\delta t}T(x,t)=a*\frac{\delta^2}{\delta x^2}T(x,t)$

with the boundary condition: $-\lambda*\frac{\delta}{\delta x}T(x=0,t)=q(t)=q_0*\exp(-b*t)$

The solution is best obtained using the Laplace transform:

$$s*L(T(x,t),t,s)-T(x,0)=a*(\frac{\delta^2}{\delta x^2}L(T(x,t),t,s))$$

If we set T(x,0)=0, the solution is

$$U(x,s)=\exp(\frac{\sqrt{(s)}*x}{\sqrt{(a)}})*F_2(s)+\exp(\frac{-\sqrt{(s)}*x}{\sqrt{(a)}})*F_1(s)$$

Since the solution must be bounded as x approaches infinity, it follows that F_2 (s) = 0 .

The transformed boundary condition gives $-\lambda*\frac{\delta}{\delta x}U(x,s)=\frac{q_0}{s+b}$.

The derivative of the solution U(x,s) yields: $-\lambda*\frac{\delta}{\delta x}\exp(\frac{-\sqrt{(s)}*x}{\sqrt{(a)}})*F_1(s)=\frac{q_0}{s+b}$

It follows at the boundary x = 0 that : $F_1(s)=\frac{q_0*\exp(0)*\sqrt{(a)}}{\sqrt{(s)}*\lambda*(s+b)}$

Thus, the Laplace transform of the solution is: : $U(x,s)=(\frac{q_0*\sqrt{(a)}}{\sqrt{(s)}*\lambda*(s+b)})*\exp(\frac{-\sqrt{(s)}*x}{\sqrt{(a)}})$

With the abbreviations $C=(q_0*\sqrt{a}/\lambda)$ and $k=x/\sqrt{(a)}$ we get $U(x,s)=C*\frac{\exp(-\sqrt{(s)}*k)}{\sqrt{(s)}*(s+b)}$

To find the inverse Laplace transform of $F(s)=\frac{\exp(-k*\sqrt{(s)})}{(s+b)*\sqrt{(s)}}$

we use the property of convolution and known correspondences from the Laplace table.

Step 1: Identification of the components

We can write F(s) as the product of two functions G(s) and H(s)

$$G(s)=\frac{\exp(-k*\sqrt{s})}{\sqrt{s}} \quad \text{und} \quad H(s)=\frac{1}{s+b}$$

Step 2: Inverse Laplace transform of the individual parts

From the standard tables of the Laplace transform, we know:

$$L^{-1}(G(s))=L^{-1}(\frac{\exp(-k*\sqrt{s})}{\sqrt{s}})=\frac{1}{\sqrt{\pi*t}}*\exp(\frac{-k^2}{4*t})=g(t)$$

$$L^{-1}(H(s))=L^{-1}(\frac{1}{s+b})=\exp(-b*t)=h(t)$$

Step 3: Application of the convolution theorem

The convolution theorem states: $L^{-1}(G(s)*H(s))=(g*h)(t)=\int_0^t g(\tau)*h(t-\tau)\,d\tau$

Let us substitute our functions:

$$f(t)=(g*h)(t)=\int_0^t \frac{1}{\sqrt{(\pi*\tau)}}*\exp(-\frac{k^2}{4*\tau})*\exp(-b*(t-\tau))\,d\tau$$

Let us take the constant $\exp(-b*t)$ from the integral:

$$f(t)=\frac{\exp(-b*t)}{\sqrt{(\pi)}}*\int_0^t \frac{1}{\sqrt{(\tau)}}*\exp(-\frac{k^2}{4*\tau}+b*\tau)\,d\tau$$

Step 4: Solving the Integral

To solve this integral, a substitution is made (typically $u=\sqrt{(\tau)}$, with $du=\frac{1}{2*\sqrt{(\tau)}}d\tau$)

$$f(t)=\frac{2*\exp(-b*t)}{\sqrt{(\pi)}}*\int_0^{\sqrt{t}}\exp(-\frac{k^2}{4*u^2}+b*u^2)\,du \quad .$$

The exponent in the integral can be expanded

$$-\frac{k^2}{4*u^2}+b*u^2=-(\frac{k}{2*u}+\sqrt{b}*u)*(\frac{k}{2*u}-\sqrt{b}*u)=-(\frac{k}{2*\sqrt{(\tau)}}+\sqrt{b*\tau})*(\frac{k}{2*\sqrt{(\tau)}}-\sqrt{b*\tau})$$

After applying the standard integrals with partial integration for this specific form, the result is obtained using the error function erf(z) or the complementary error function erfc(z).
CAUTION when using artificial intelligence. 3 attempts yielded 3 different incorrect results.

The result, using the abbreviations $y=x*\sqrt{(b/a)}$ and $\xi=x/\sqrt{(4*a*t)}$, is :

$$f(t,x)=\frac{-I*(\exp(-b*t))}{(2*\sqrt{(b)})}*(\exp(-I*y)*erfc(\xi-I*\sqrt{(b*t)})-\exp(I*y)*erfc(\xi+I*\sqrt{(b*t)}))$$

The entire function $u(s,x)=\frac{q_0*\sqrt{(a)}}{\lambda}*\exp\frac{(-\sqrt{(s/a)}*x)}{(\sqrt{(s)}*(s+b))}$ is transformed back to

$$T(t,x)=\frac{-I*q_0*\exp(-b*t)}{2*\sqrt{(b/a)}*\lambda}*(\exp(-I*y)*erfc(\xi-I*\sqrt{b*t})-\exp(I*y)*erfc(\xi+I*\sqrt{b*t}))$$

where $y=x*\sqrt{(b/a)}$ and $\xi=x/\sqrt{(4*a*t)}$

The material properties are:

Decay rate	$b = 4{,}588 * 10^{-10} 1/s$	Density of the salt	$\rho = 2200\ kg/m^3$
Specific heat capacity	$c_p = 1200\ J/(kg * K)$	Thermal conductivity	$\lambda = 5{,}4\ W/(m * K)$
Thermal conductivity	$a = \lambda/(\rho * c_p) = 2{,}045454546 * 10^{-6}\ m^2/s$		

Thus, the spatial and temporal temperature distribution is given by:

$$T(x,t) = -I * 7{,}71642253 * \exp(-4{,}588 * 10^{(-10)} * t)$$
$$*(\exp(-I * 0{,}01497672268 * x) * erfc(349{,}6029494 * x/\sqrt{(t)} - I * 0{,}2141961718 * 10^{-4} * \sqrt{(t)})$$
$$- \exp(I * 0{,}01497672268 * x) * erfc(349{,}6029494 * x/\sqrt{(t)} + I * 0{,}2141961718 * 10^{-4} * \sqrt{(t)}) \ )$$

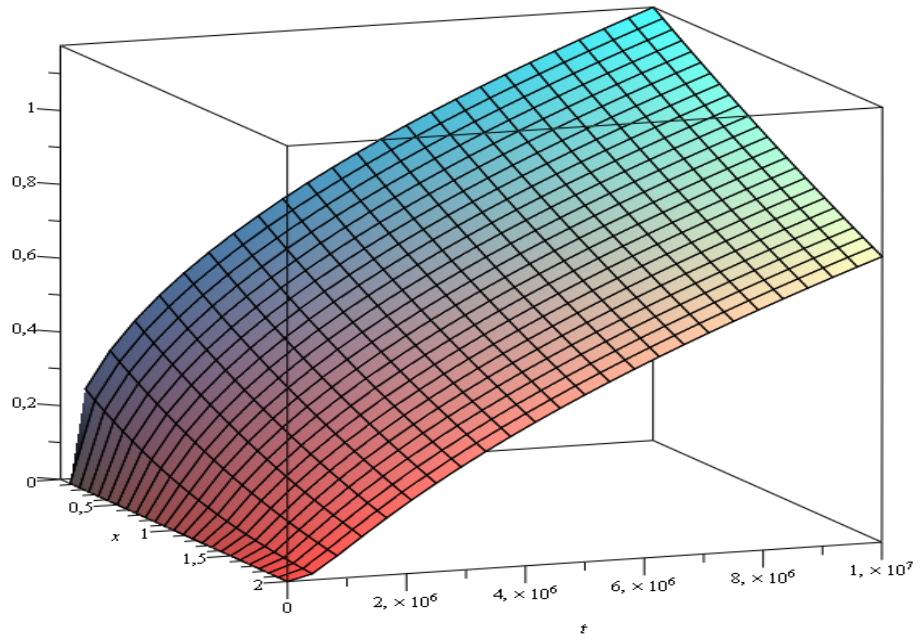

Fig. 1 Temperature field in K, Distance x in m, time t in s t < 115 days

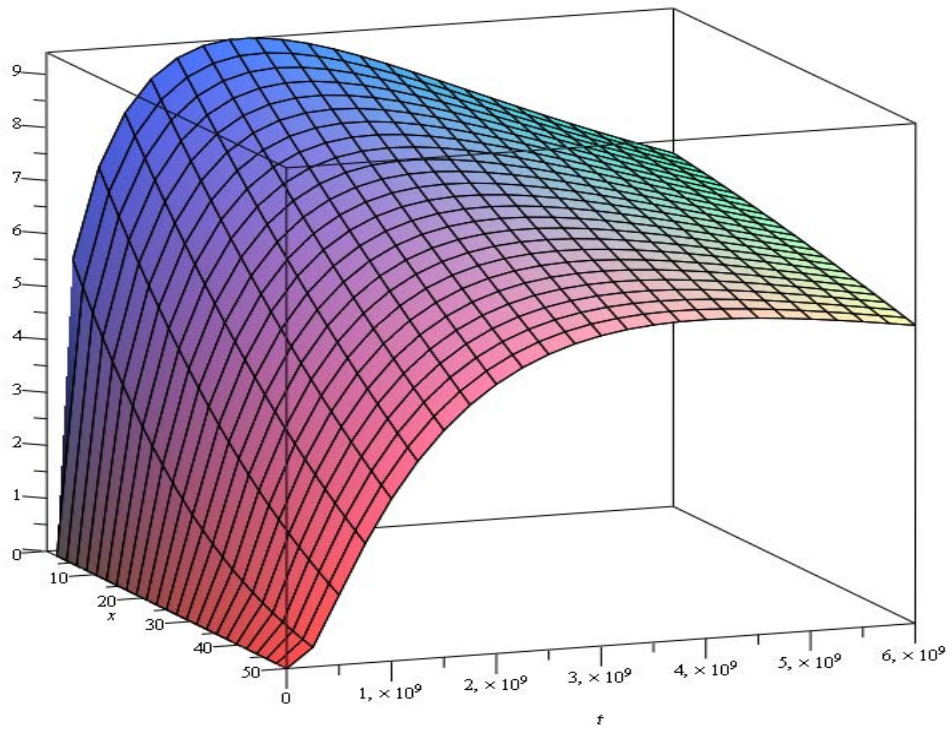

Fig. 2 Temperature field in K, Distance x in m, time t in s t < 190 years

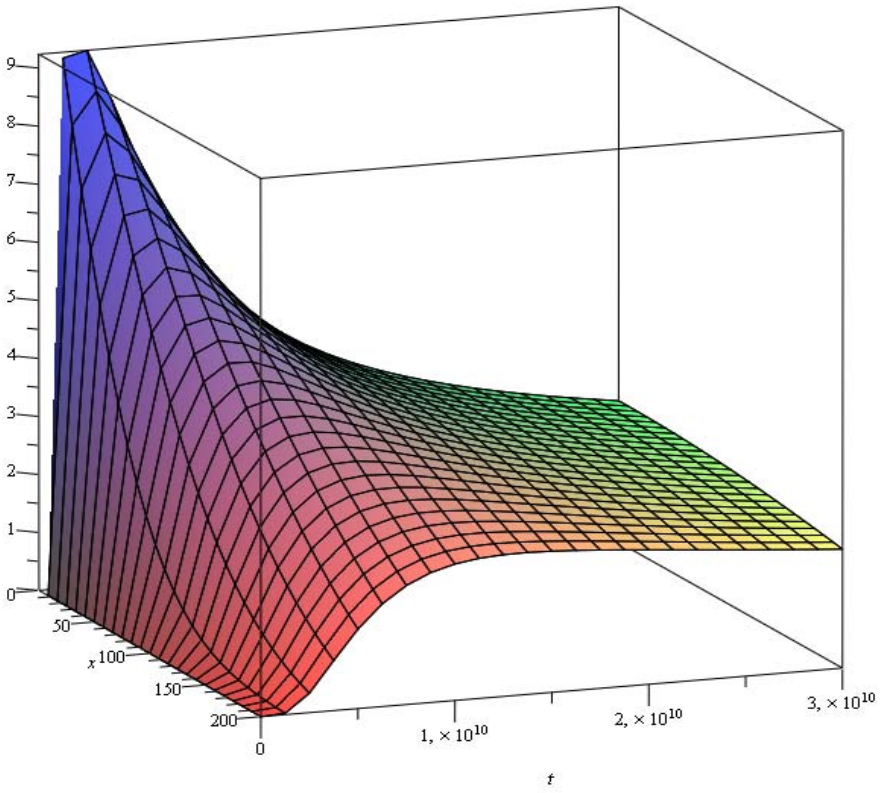

Fig. 3 Temperature field in K, Distance x in m, time t in s t < 950 years

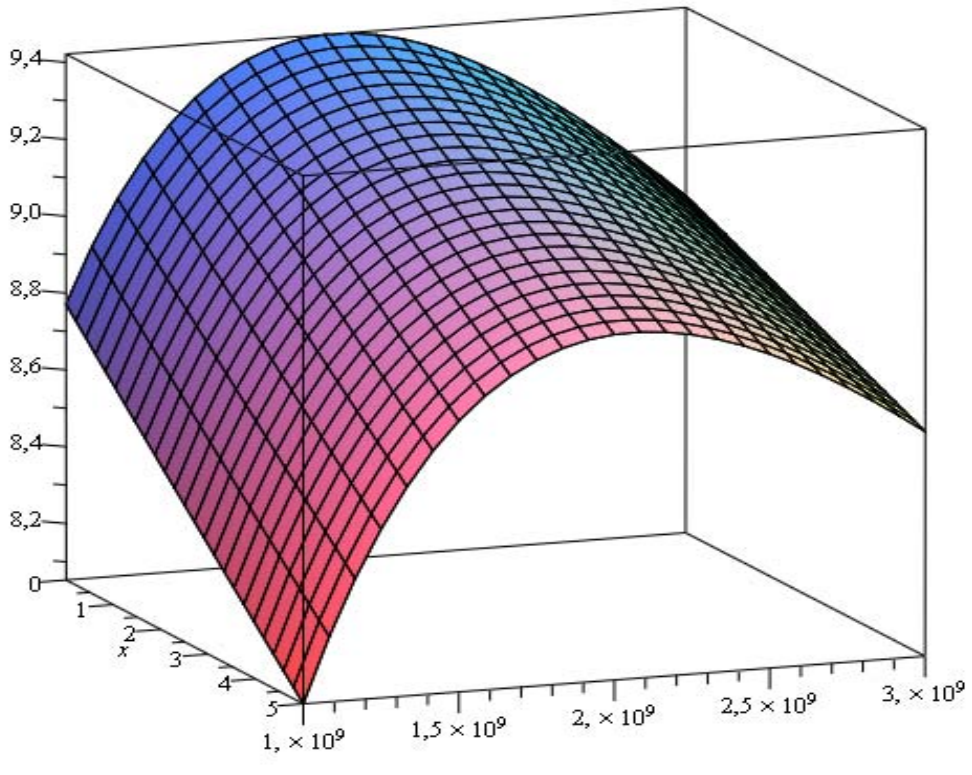

Fig. 4 Temperature field in K, Distance x in m, time t in s, maximum at 9,4 K after 58.6 years

The maximum temperature of 9.4 K is reached after approximately 58.6 years at the level of the containers (topmost curve). The curves below represent greater distances of up to 5 m from this level.

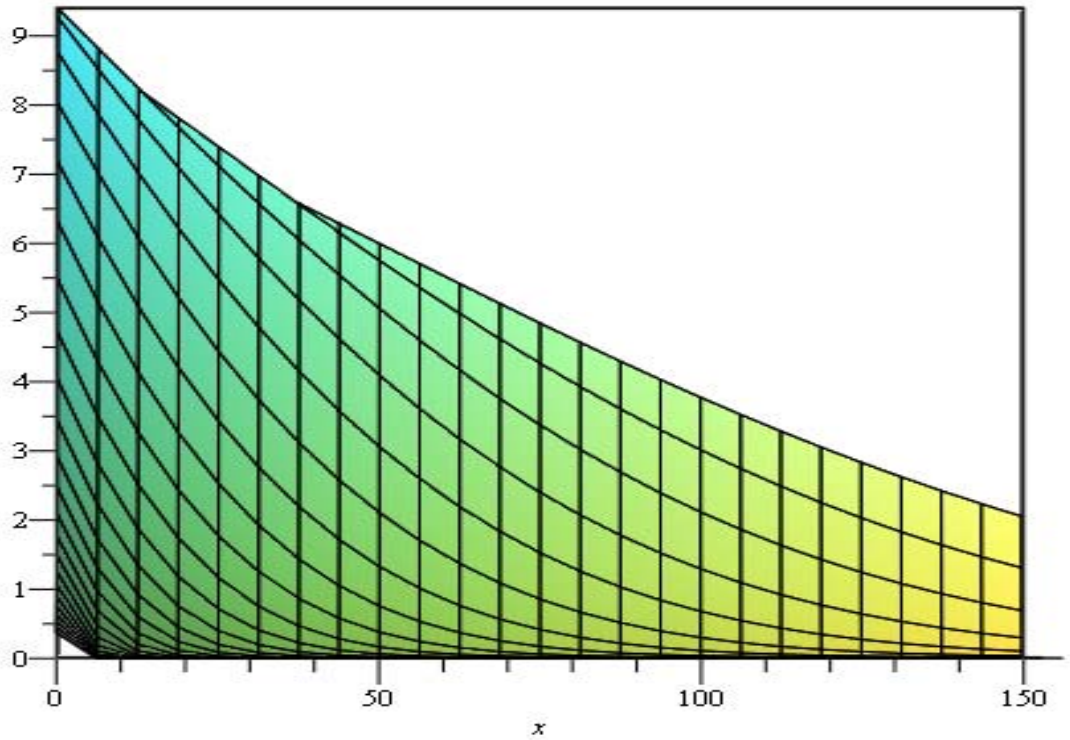

Fig. 5 Temperature field in K, Distance x in m, time t in s, 10^6 s < t < $10^{9.6}$ s = 126.2 years

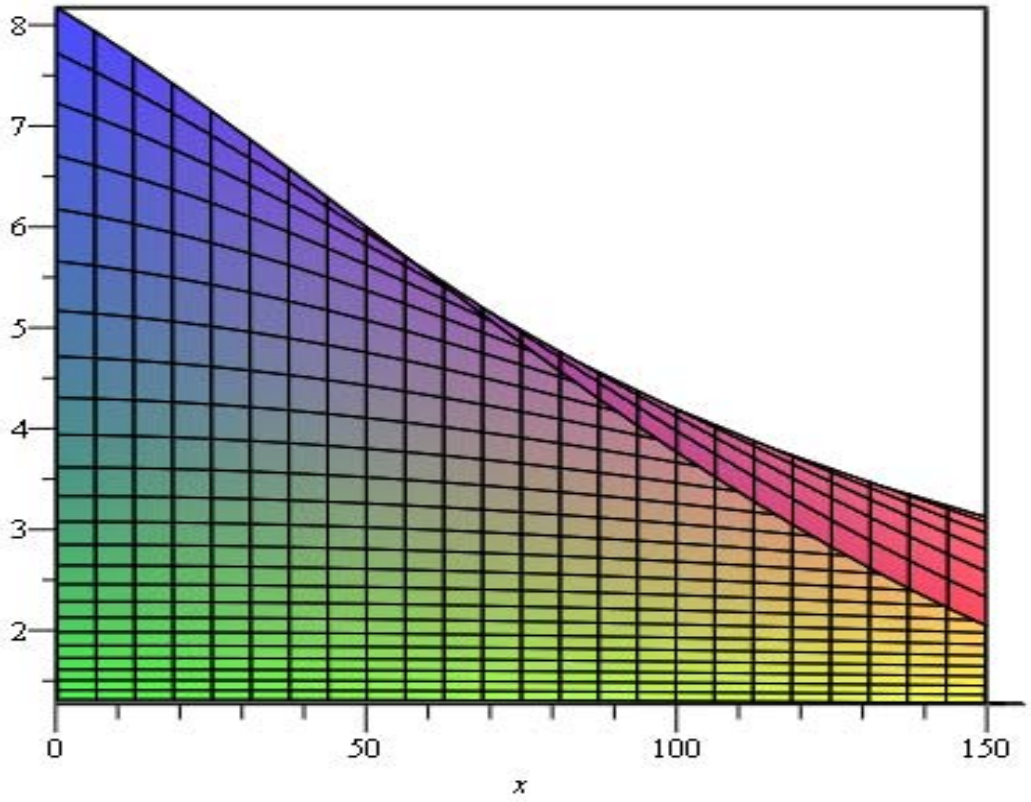

Fig. 6 Temperature field in K, Distance x in m, time t in s, $10^{9.6}$ s < t < 10^{11} s = 3170 years

The temperature rises slowly at first until it reaches its maximum after 58.6 years; the heat then penetrates further and further into the salt, and the temperature falls again. After 950 years, the heat is distributed almost evenly over the next 500 m, and the temperature is approximately 2.4 K at the heating level and just under 1.3 K at a distance of 500 m.

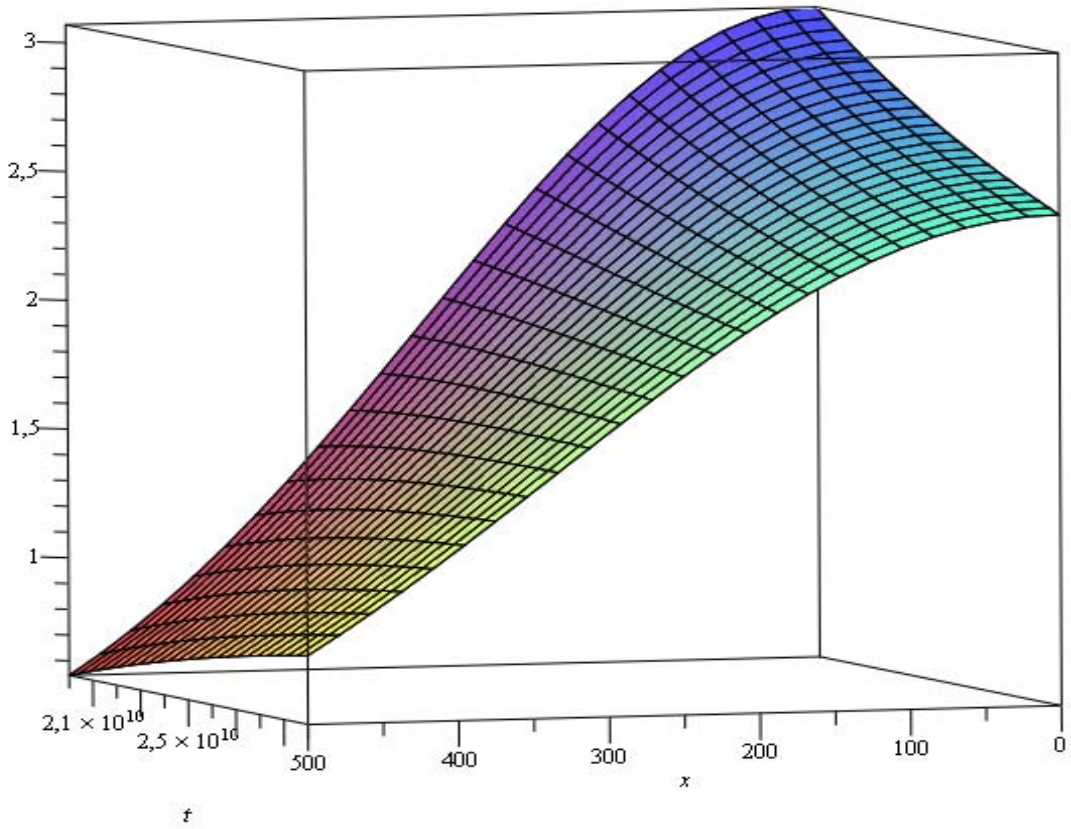

Fig. 7 Temperature field in K, Distance x in m, time t in s, $t < 950$ years

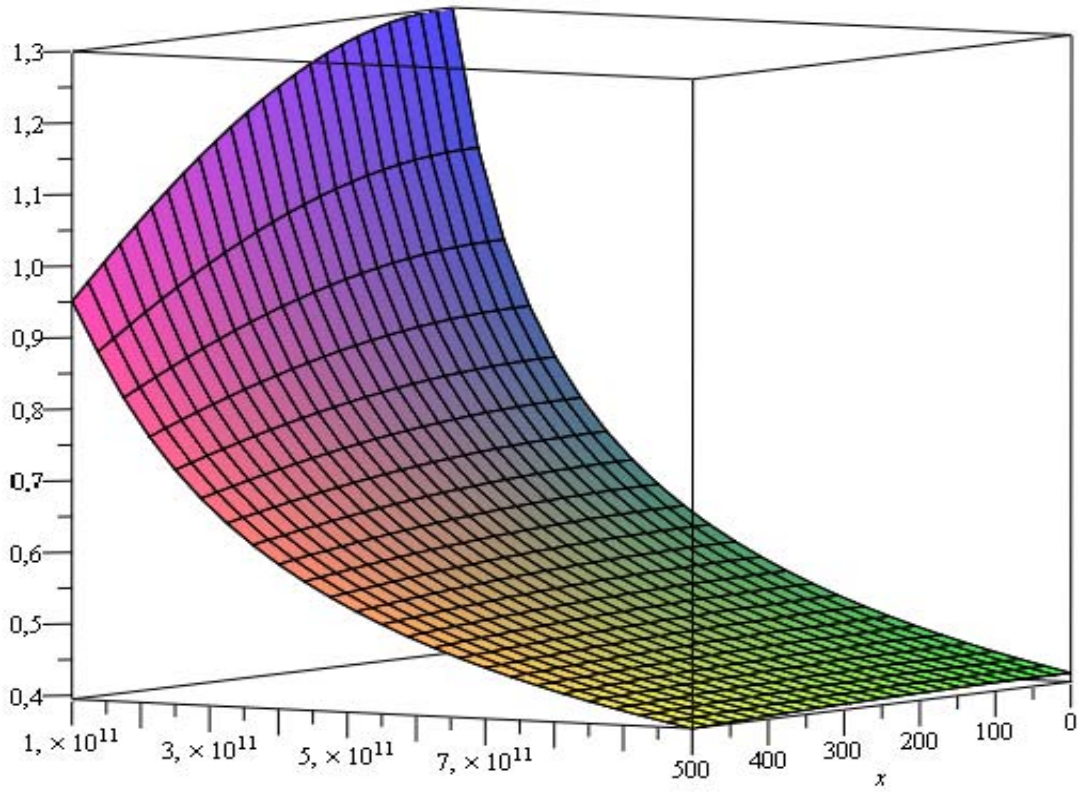

Fig. 8 Temperature field in K, Distance $x < 500$ m, time t in s, $t < 10^{12}$ s = 31688 years

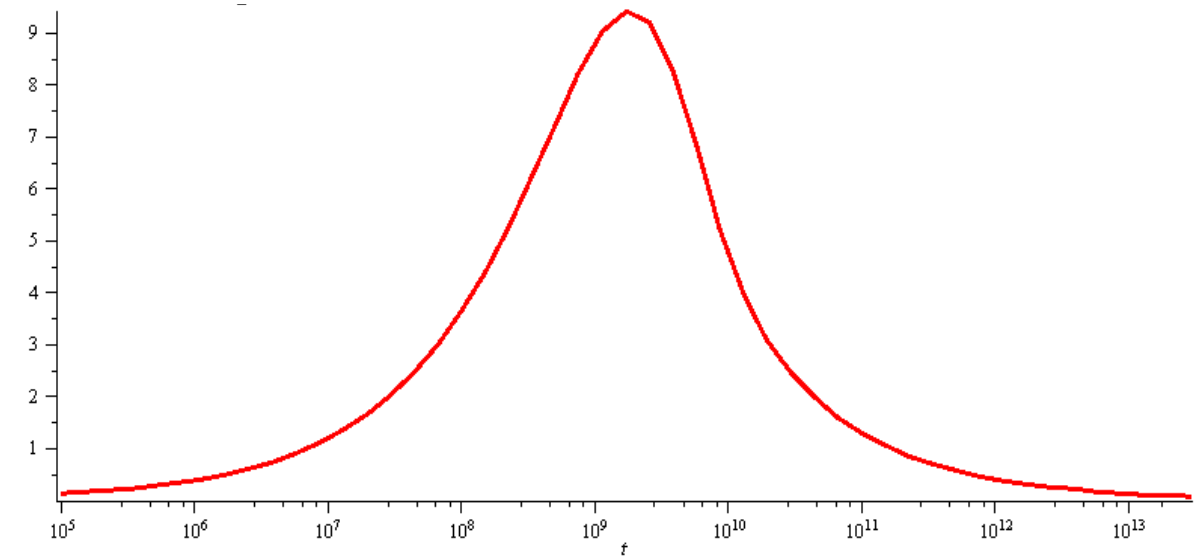

Fig. 9 Temperature at container layer in K, time 10^5 s = 1,16 days < t < $3*10^{13}$ s = 950642 years, peak after $1.85*10^9$ s = 58.6 years

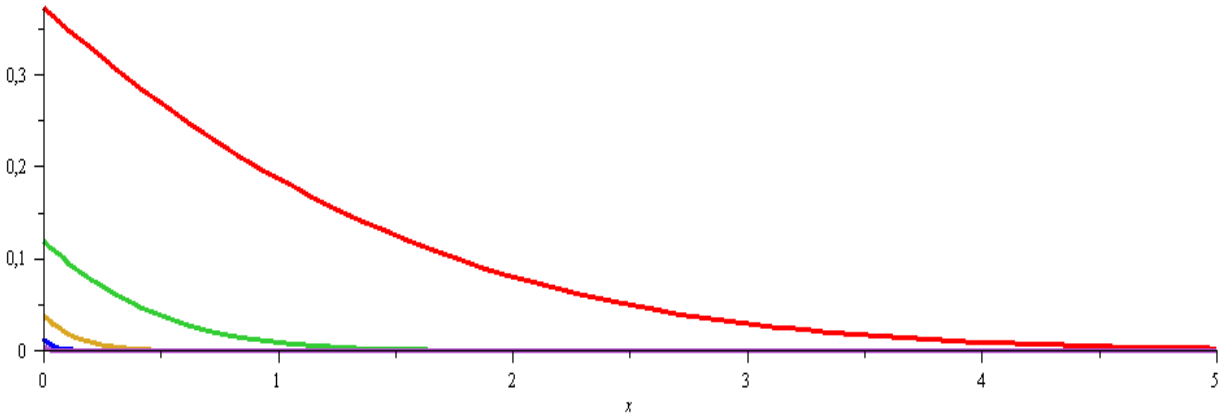

Fig. 10 Temperature in K at distance x < 5 m, time t = 1000 s, 10^4 s, 10^5 s, 10^6 s=11.6 days

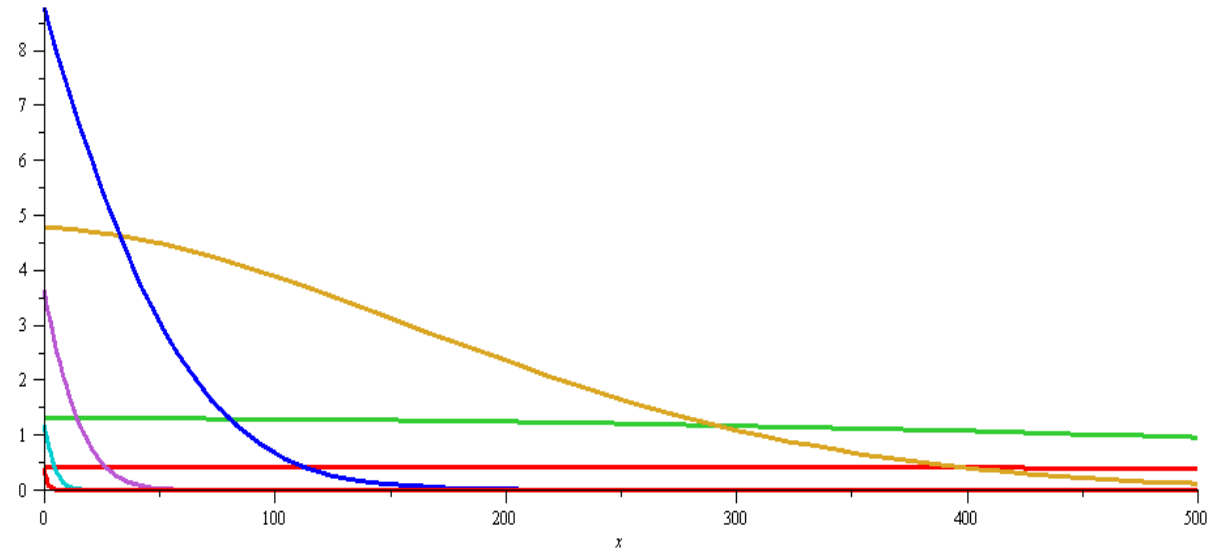

Fig. 11 Temperature in K at distance x < 500 m,
time t = 10^6 s, 10^7 s, 10^8 s, 10^9 s, 10^{10} s, 10^{11} s, 10^{12} s = 31688 years

The incoming heat causes the temperature to rise slowly until, after approximately 10^9s =31,7 years, the heat input is too low to maintain the temperature in the heating plane. The temperature drops again and, after 10^{12} s = 31688 years, the heat is evenly distributed and the temperature is less than 0.5 K higher everywhere than at the start.